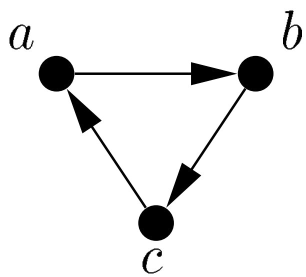
The Variety Generated by Tournaments

Miklós Maróti
Vanderbilt University

Tournaments

Definition. A *tournament* is a complete directed graphs with loops



$$xy = yx = x \text{ if and only if } x \to y$$

	a	b	c
a	a	a	c
b	a	b	b
c	c	b	c

a commutative groupoid satisfying $xy \in \{x, y\}$ (*conservative law*).

Simple tournaments

Theorem (P. Erdős, E. Fried, A. Hajnal, E.C. Milner). *Every tournament (with the exception of odd chains) has a one-point extension to a simple tournament.*

Congruences and automorphisms

Definition. A finite lattice is *admissible* if it is isomorphic to the congruence lattice of a tournament.

Theorem (V. Müller, J. Nešetřil, J. Pelant). *Given an admissible lattice* $\mathbf{L}$*, an odd group* $\mathbf{G}$*, and a tournament* $\mathbf{T}$*, there is a tournament* $\mathbf{A}$ *such that* $\operatorname{Con}\mathbf{A} \cong \mathbf{L}$, $\operatorname{Aut}\mathbf{A} \cong \mathbf{G}$ *and* $\mathbf{T} \leq \mathbf{A}$.

- Admissible lattices are charaterized.
- Automorphism groups of tournaments are groups of odd order (**J.W. Moon**).

Equations of tournaments

Theorem (J. Ježek, P. Marković, M. Maróti, R. McKenzie). *The following four equations form a base for the 3-variable equations of tournaments:*

(1) $xx = x$

(2) $xy = yx$

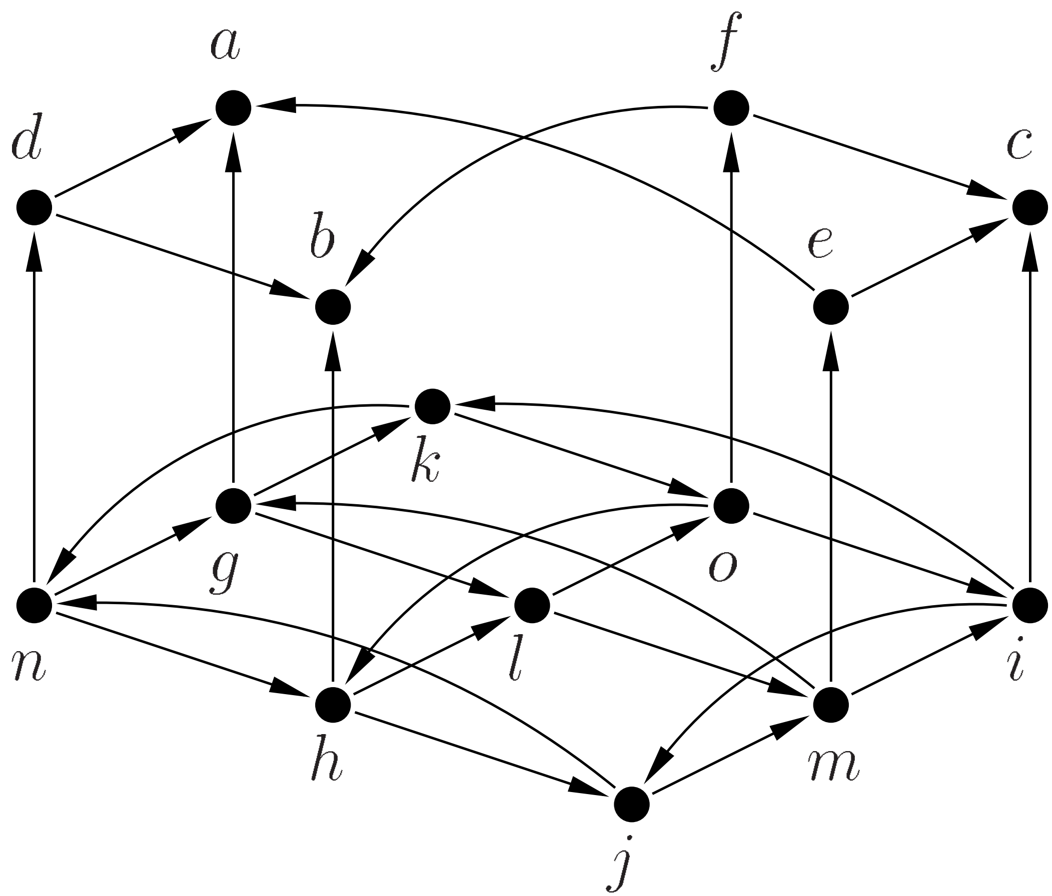
(3) $(xy)x = xy$

(4) $(xy \cdot xz)(xy \cdot yz) = (xy)z$

Question (V. Müller, J. Nešetřil, J. Pelant). Are tournaments finitely based?

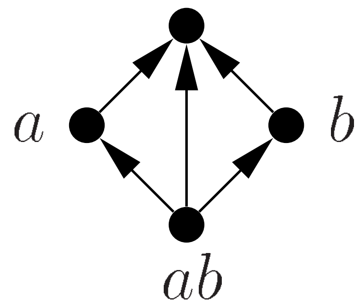
Definition. Denote by $\mathcal{T}$ the variety generated by tournaments.

The 3-generated free algebra in $\mathcal{T}$



The variety generated by tournaments

Not every member of $\mathcal{T}$ is a tournament. We write $x \to y$ if $xy = x$.



Theorem (J. Ježek, P. Marković, M. Maróti, R. McKenzie).
The variety $\mathcal{T}$ is

(1) *locally finite* (**S. Crvenković, I. Dolinka, P. Marković**),

(2) *not finitely based,*

(3) *inherently non-finitely generated,*

(4) *congruence meet-semidistributive.*

Subdirectly irreducible algebras

Question (R. McKenzie). Are all subdirectly irreducible members of $\mathcal{T}$ tournaments?

Theorem (R. McKenzie). *All simple members of $\mathcal{T}$ are tournaments.*

Definition. We call an algebra $\mathbf{A} \in \mathcal{T}$ *strongly connected* if for any $a, b \in A$ there exists a path $a = a_0 \to a_1 \to \cdots \to a_{n-1} = b$.

Lemma (J. Ježek). *If all strongly connected, subdirectly irreducible members of $\mathcal{T}$ are tournaments, then all subdirectly irreducibles are tournaments.*

The proof

- minimal representation of $\mathbf{S}$
- weakly indecomposable subdirect products
- triangular graphs
- maximal spanning triangular graphs
- subdirect products of finitely many strongly connected tournaments are strongly connected
- triangular algebras
- unique coatom β in $\operatorname{Con}\mathbf{S}$
- the blocks of β, the arrows
- blow-up composition

Consequences

Corollary. *Every finitely generated subvariety of var*T *has a finite residual bound.*

Corollary. *Every finitely generated subvariety of* $\mathcal{T}$ *is finitely based.*

Corollary. *The lattice of subvarieties of* $\mathcal{T}$ *is distributive. Every join irreducible subvariety of* $\mathcal{T}$ *is generated by a unique (up to isomorphism), hereditarily zeroless finite subdirectly irreducible tournament.*

Corollary. *The lattice* 2^ω *can be embedded into the lattice of subvarieties of* $\mathcal{T}$.